Model Theory – Lecture 6 – types and saturated models 1

New good reference as notes check on the website!

<u>Definition</u> Let (T be a theory), $\mathcal{M}$ a model and a a tuple in $|\mathcal{M}|^n$. The "type of a", $tp_{\mathcal{M}}(a)$, is the set of formulas in the language that a makes true $\mathcal{M} \models \varphi(x/a)$

↳ not necessary, $\mathcal{M}$ would be a structure

Example $\mathcal{L}_F$ is the language of fields, $\mathcal{M} = \mathbb{C}$, $a = \sqrt{2}$

↳ complex numbers

Then, $\varphi(x) \equiv x^2 - 2 \in tp_{\mathbb{C}}(a)$ and $\varphi(x) \in tp_{\mathbb{C}}(-a)$,

so a formula can belong to different types.

Warning! Actually $tp_{\mathbb{C}}(\sqrt{2}) = tp_{\mathbb{C}}(-\sqrt{2})$ Indeed,

$\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(x) = -x$ is an automorphism of the structure

<u>Question</u> If φ is an automorphism, then $tp_{\mathcal{M}}(x) = tp_{\mathcal{M}}(\varphi(x))$ Is the converse true?

Answer Yes, up to enlarging the model (saturated models)

This is similar to what happens in Galois theory

For example, consider $\pi \in \mathbb{C}$ Then, $tp_{\mathbb{Q}}(\pi) \ni p(x) \neq 0$ for every polynomial with integer coefficients The same formulas would live in $tp_{\mathbb{Q}}(e)$ and so every other trascendental number.

We now make an example in the language of ordered fields $\mathcal{L}_{OF}$
In $M = \mathbb{R}$, clearly $tp_{\mathbb{R}}(\sqrt{2}) \neq tp(-\sqrt{2})$ For example $x<0$ lives in the latter and not in the former

<u>Question</u> Consider $(\mathbb{Q}, +, 0, <)$, how many types? Three $x<0$, $x=0$, $x>0$

One can use affinities

what if we add 1 to the language? There are infinite types

Definition Let M be a structure in a language $\mathcal{L}$. A "type in M" is a set of formulas Γ such that it is finitely satisfiable, i.e., for every $\Gamma_0 \subseteq \Gamma$ finite, there exists a tuple $a \in |M|^n$ such that $M \models \bigwedge_{\varphi \in \Gamma_0} \varphi(a)$

Definition: A type in M is "realized" if a in the previous definition satisfies $M \models \bigwedge_{\varphi \in \Gamma} \varphi(a)$

A type is "complete" if for all formulas ψ or $\neg\psi$ are in it

Notation an n-type is a type whose formulas have n variables

Example Consider $(\mathbb{Q}, +, \cdot, \leq)$ We prove that uncountably many types are <u>not</u> realised Choose a real number r

Consider $\varphi_q(x) \equiv x > q$ with $q < r$ (notice we can't say this yet, as r is not here, also if q is $\frac{2}{5}$ we would write really $5x < 2$) Then $\{\varphi_q(x)\}_{q<r}$ is a type and doesn't have a global satisfier

Notice it would be fairly easy to add such a global satisfier

<u>Definition</u> Let Φ be a theory A "(n-) type" of Φ is a collection of formulas (in n variables) Γ such that $\Phi \cup \Gamma$ is finitely satisfiable (Ivan also uses "coherent" sometimes)

<u>Proposition</u>: Every type in the theory of a model is realised in some extension of the model

<u>Proof</u> We use ultraproducts to construct the model and get the element that satisfies the type by compactness

Theorem: the space of types over a theory is compact (and T_2)

Proof: Let Φ be a theory. The "space of types" has as an underlying set the union of the complete n-types of Φ for every n

The topology is the one given by the basis β

for every $\varphi(x)$, $V_{\varphi(x)} = \{ t \mid \varphi(x) \in t \in \bigcup_{n \in \mathbb{N}} S_n(\Phi) \} \in \beta$

We show it is T_2. Let t_1 and t_2 be different types

There is a formula $\varphi \in t_1 \setminus t_2$. Then $V_\varphi \ni t_1$ and $V_{\neg\varphi} \ni t_2$, and

$V_\varphi \cap V_{\neg\varphi} = \emptyset$. That is to say, it is Hausdorff

Let $\mathcal{U}_\alpha$ be an open cover of the space. We can reduce to the case in which all such open sets are basic sets

By absurd. Let $\mathcal{F}$ be a finite subfamily of $\mathcal{U}_\alpha$, then they do not cover the space. Let $\Phi \cup \{\neg\varphi\}_{\varphi \in \mathcal{F}}$. This is a type and it does not live in $\bigcup \mathcal{U}_\alpha$

∎

here we really mean that V_φ is in $\mathcal{F}$ and since they are all basic open sets this is a fair abuse of notation